Steady Magnetohydrodynamic Flow Past a Nonconducting Wedge

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This paper presents a study of the steady two-dimensional magnetohydrodynamic flow of an infinitely conducting fluid past a nonconducting wedge with nonaligned flow and magnetic field. The flows considered are in the "superfast" or fully hyperbolic regime. The flows consist of several regions of uniformity connected by shocks and expansion waves. Because of the boundary condition on the magnetic field, the magnetic field must be the same in the regions above and below the wedge; thus the flows in these regions are coupled, unlike in the case of ordinary supersonic gasdynamics. Only small wedge angles and weak waves (characteristics) are considered. The problem thus is linearized, and explicit solutions are obtained which are qualitatively similar to the nonlinear solutions. Some interesting and unexpected features arise, and they are discussed in detail.

1. Introduction

ONE of the simplest and most fundamental problems in two-dimensional supersonic aerodynamics is the flow past a wedge or, equivalently, flow past a sudden corner. It is well known that, if the flow is deflected in one direction, a centered rarefaction wave (Prandtl-Meyer expansion) results, whereas if the flow is deflected in the opposite direction, a plane oblique shock results, provided that the angle of deflection is not too large (Fig. 1). In either case, regions of uniformity are joined together by a "wave," a phenomenon possible only in supersonic flow where the differential equations are hyperbolic. In subsonic flow, where the differential equations are elliptic, it is never possible to have regions of uniformity unless the entire flow is uniform.

The analogous problem in magnetohydrodynamics is considered here. Let an inviscid, infinitely conducting fluid at a speed greater than the "fast speed" of the fluid (corresponding to the magnetic field) flow past a wedge or a corner, so that the flow differential equations are fully hyperbolic. Let the material constituting the wedge or walls be a perfect nonconductor. If the material is a perfect conductor, it is well known that no flow is possible at all (see, e.g., Grad[5]). For a nonconducting wall, the phenomenon almost would be expected to be the same as in ordinary aerodynamics, with several waves, perhaps, instead of each one. Surprisingly, this is true only in the wedge problem but not in the corner problem; in magnetohydrodynamics, the two problems are not equivalent. In fact, as will be seen, there cannot be a region of uniformity anywhere in the corner flow. In the wedge flow problem, there are also some phenomena that are "unexpected" from gasdynamics alone.

A particularly simple and degenerate case in magnetohydrodynamics is that of the aligned field, i.e., the magnetic field and velocity are everywhere parallel. In this case, the boundary conditions simplify, and the corner and wedge flows are identical; there is complete analogy with supersonic aerodynamics. Moreover, as in ordinary aerodynamics, a single shock or expansion wave suffices to produce a deflection around a corner, although this wave may point upstream. A complete solution of this problem has been given by Bazer and Ericson.[1]

For nonaligned fields, the treatment of a single shock has been discussed by Bazer and Ericson[1] and by Kiselev and Kolosnitsyn.[7] The latter also suggested that the results apply immediately to corner flow, but this is incorrect, since the boundary conditions on the magnetic field have not been imposed properly.

In this paper, the analysis is limited to small-angled wedges, so that a linear theory holds. The analysis necessary for the description of the flow is quite simple. The major aim is to illustrate the effect of the coupling of the flows above and below the wedge through the requirement of the magnetic boundary conditions. The results so obtained, although not quantitatively valid for large-angled wedges, nevertheless give a physical picture that is qualitatively correct. Needless to say, the entire families of strong shocks (fast and slow) are excluded from consideration, and such questions as maximum angle of deviation of the fluid before the shock detaches, etc., cannot be answered until results are available from a nonlinear analysis. These will be reported in a subsequent paper.

2. Formulation of the Problem

The wedge problem is now formulated, and it will be seen that the flow can be constructed from several uniform flows in different regions. Concurrently, it also will be indicated that the corner problem cannot be so treated.

Consider a perfectly insulating two-dimensional wedge (Fig. 2) with wedge angles θ and θ', as shown. Let a uniform stream of inviscid infinitely conducting fluid flow toward it with a velocity U, and let there be a uniform magnetic field B_0 inclined at angle ψ_0 to the velocity ($\psi_0 \neq 0$). It is assumed that U is sufficiently high so that the flow differential equations in this uniform region 0 are hyperbolic. The flow is to turn the angles through oblique (plane) shocks and/or centered expansion (plane) waves into the uniform regions 2 and 2′. It will be shown that such a flow is possible.

For simplicity, let all the flow variables lie in the two-dimensional x-y plane. This is sometimes called a restricted two-dimensional problem, in contradistinction to a general two-dimensional problem, in which the dependent variables

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are functions of two space variables but possess three components. The extension to this latter case is rather straightforward, as will be indicated later, and so the simpler case will be considered here.

At each point in a restricted two-dimensional flow, there can be only two weak oblique shocks, one "fast" and one "slow." This can be seen readily from the jump conditions or simply by comparison with one-dimensional unsteady theory. Thus, if a flow of the desired type is to exist, there can be only two plane shocks or expansion waves, which separate the regions 0 and 2 on top (or regions 0 and 2′ on the bottom) and which include another uniform region 1 (or 1′).

Should one wish to consider the general two-dimensional problem, one merely would have three shocks (or expansion waves) on top and three on bottom, since now there will be Alfven waves, which do not occur in the restricted two-dimensional problem. All other considerations are the same.

To see the correct formulation of the problem, one must resort to counting. The dependent variables are pressure p, density ρ, velocity $\bar{q}$ (components u,v in x,y directions), and magnetic field $\bar{B}$ (components B_x, B_y), six variables in each region. For the four regions 1, 2, 1′, 2′, this makes 24 unknowns; all quantities in region 0 are, of course, given. In addition, there are the four shock angles ω_f, ω_s, ω_f', ω_s' to be found (or for expansion waves, the angles of the last plane wave). Thus the total number of unknowns in this problem is 28.

Each shock (or expansion wave) provides six equations connecting the variables on each side. In addition, the boundary conditions in downstream are 1) the velocity components in region 2 must be tangential to the wedge; 2) the velocity components in region 2′ must be tangential to the wedge; and 3) the magnetic field $\bar{B}_2$ must equal the magnetic field $\bar{B}_2'$ (two equations). Thus, there is a total of $4 \times 6 + 4 = 28$ equations, and the problem is solvable provided that the speed, field, wedge angles, etc., are given in the proper ranges so that all the solutions of the algebraic equations are real (corresponding to fully hyperbolic flow). The explicit solutions will be given for the small wedge case.

The boundary condition 3, that the field $\bar{B}_2 = \bar{B}_2'$, requires some discussion. At either face of the wedge, $\bar{B}$ must be continuous. The normal component must be continuous because the divergence of $\bar{B}$ must vanish, whereas the tangential component must be continuous because, otherwise, there will occur a current sheet, which can be located neither in the solid (it is a perfect insulator) nor in the fluid (tangential stress will result, which is impossible in an inviscid fluid). Therefore, at either surface of the wedge, the value of $\bar{B}$ is constant. But, inside the solid, $\bar{B}$ satisfies div $\bar{B} = 0$ and curl $\bar{B} = 0$. Its components therefore form the real and imaginary parts of an analytic function of the variable $z = x + iy$. Hence, if it is constant on one line, it is constant throughout. Thus $\bar{B}$ on the two surfaces must be identical.

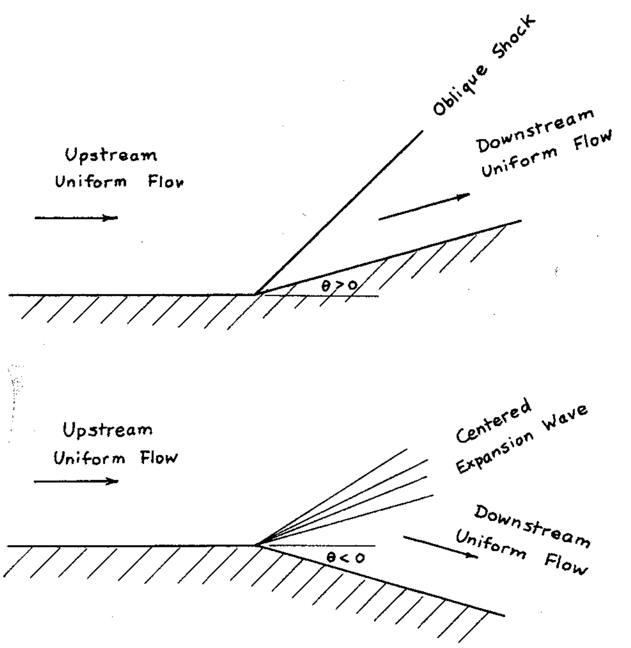

Fig. 1 Flows around corners in ordinary supersonic gasdynamics

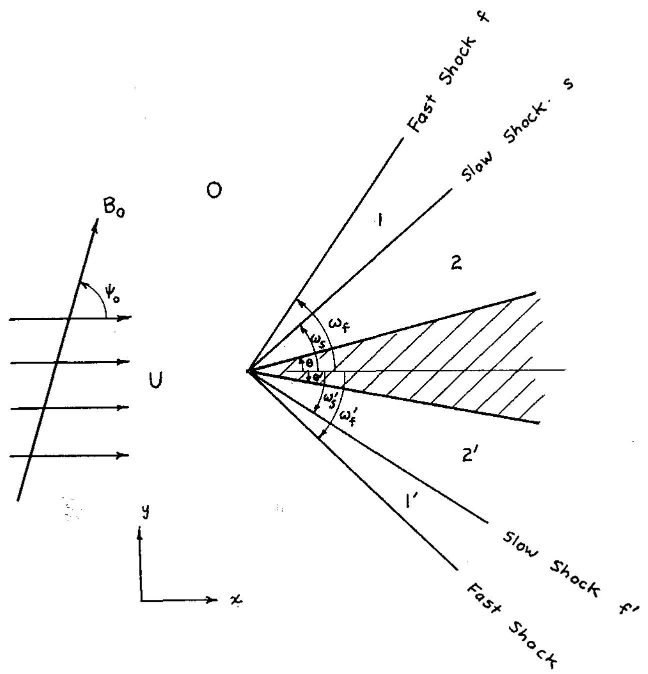

Fig. 2 Super-fast magnetohydrodynamic flow past a nonconducting wedge (as shown, θ, ω_f, $\omega_s > 0$; θ', ω_f', $\omega_s' < 0$; only shocks are shown for simplicity, although expansion fans may occur instead)

Now consider the corresponding case of flow around a corner (Fig. 3). By exactly the same reasoning as before, there will be a total of 14 unknowns. Again there will be six equations per shock, plus boundary conditions, which are 1) velocity downstream is tangential to the surface; and 2) magnetic field downstream equals magnetic field upstream (two equations). Thus, the total number of equations is 15! Consequently, such a flow, consisting of two uniform regions joined by two waves, cannot occur. Physically, the explanation is quite clear also. To have a uniform region upstream requires that disturbances do not travel upstream, a feature of flows governed by hyperbolic equations. However, in this case, magnetic disturbances are being fed back upstream through the insulating solid; hence the upstream and, for that matter, the downstream also are both disturbed and cannot remain regions of uniform flow. Kiselev and Kolosnitsyn[7] erred in that they completely ignored the magnetic field in the solid nonconductor.

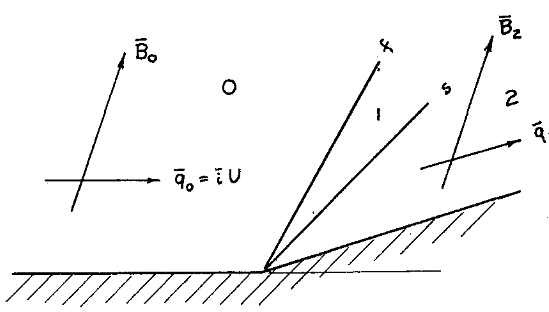

Fig. 3 Super-fast magnetohydrodynamic corner flow; incorrect formulation

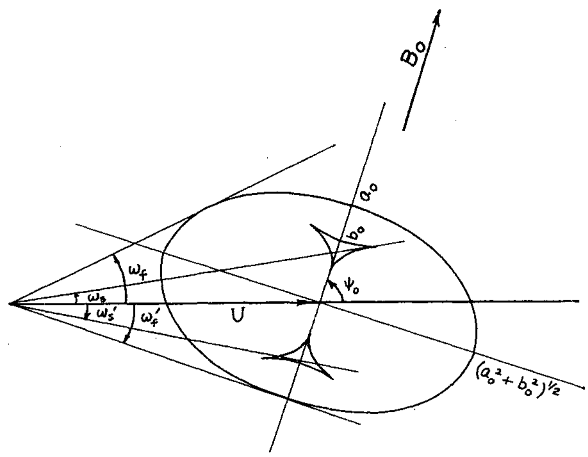

Fig. 4 Steady-flow characteristics as related to the geometry of a pulse ($\omega_f > \omega_s > 0$, $\omega_f' < \omega_s' < 0$)

3. Basic Equations

Both shocks and expansion waves may appear in a magnetohydrodynamic flow past a wedge. For the general case of a wedge with a large angle, both the jump conditions across a shock obtained from the conservation laws (see, e.g., de Hoffmann and Teller,[2] Helfer,[6] Lüst,[8] and Friedrichs[3]) and the invariants across each family obtained from the basic differential equations (see, e.g., Lyubimov,[10] Golitsyn,[4] and Lynn[9]) must be used. This discussion is limited only to flows past wedges of small angles, θ and $\theta' \ll 1$ (Fig. 2). Let $\delta(\ )$ denote the (infinitesimal) jump of a variable across a wave front. The physical state then is perturbed only slightly from the oncoming flow (i.e., $\delta\rho/\rho \ll 1$, and similarly for all other variables). Each shock or expansion wave then will be represented by a characteristic, with appropriate jumps for either compression or expansion. For convenience, the entropy s, instead of the pressure, is used as a thermodynamic variable, together with the density. The relations of δB, δq, $\delta\rho$, and δs across each characteristic are given by the conservation equations and Maxwell's equations in the well-known form (see, e.g., Friedrichs[3])

$$-q_n\delta\bar{B} + \bar{B}_0\delta q_n - B_n\delta\bar{q} = 0 \tag{1a}$$

$$-\rho_0 q_n\delta\bar{q} + a_0^2\bar{n}\delta\rho + (\bar{n}/\mu)(\bar{B}_0\cdot\delta\bar{B}) - (B_n/\mu)\delta\bar{B} = 0 \tag{1b}$$

$$-q_n\delta\rho + \rho_0\delta q_n = 0 \tag{1c}$$

$$-q_n\delta s = 0 \tag{1d}$$

where a_0^2 is the square of the gasdynamic sound speed, and $\bar{n}$ is the unit normal vector on each wave pointing downstream ($\bar{n}\cdot\bar{q} > 0$).‡ From Eq. (1d), $\delta s = 0$ is obtained, since $q_n > 0$. A necessary and sufficient condition for the existence of a solution of Eqs. (1a–1c) is the vanishing of the determinant of the coefficient matrix, which yields the well-known equation of wave speed (using the notation $b_n^2 = B_n^2/\mu\rho$, $b_0^2 = B_0^2/\mu\rho$):

$$q_n^4 - (a_0^2 + b_0^2)q_n^2 + a_0^2b_n^2 = 0 \tag{2}$$

The solution of Eqs. (1a–1c) can be written as

$$\delta\bar{B} = \epsilon K\frac{q_n^2}{U^2}(\bar{B}_0 - B_n\bar{n}) \tag{3a}$$

$$\delta\bar{q} = \epsilon Kq_n\left(\frac{1}{A^2}\frac{B_n\bar{B}_0}{B_0^2} - \frac{q_n^2}{U^2}\bar{n}\right) \tag{3b}$$

$$\delta\rho = \epsilon K\rho_0\left(\frac{q_n^2}{U^2} - \frac{1}{A^2}\frac{B_n^2}{B_0^2}\right) \tag{3c}$$

where $A^2 = U^2/b_0^2$ is the freestream Alfven number squared,

‡ The normal components q_n, B_n denote $\bar{q}\cdot\bar{n}$, $\bar{B}\cdot\bar{n}$, respectively.

and $b_0^2 = B_0^2/\mu\rho_0$ is the freestream Alfven speed squared. Here $\epsilon = +1$ is defined for fast waves and $\epsilon = -1$ for slow waves. The parameter K is a measure of the strength of each wave. As is readily apparent from (3c), $K > 0$ for compression waves and $K < 0$ for expansion waves. On the upper side of the wedge, $\omega > 0$, and $\bar{n} = \bar{\imath}\sin\omega - \bar{\jmath}\cos\omega$, where $\bar{\imath}$ and $\bar{\jmath}$ are the unit vectors in the x and y directions. Then,

$$q_n = U\sin\omega \tag{4a}$$

and

$$B_n = B_0\sin(\omega - \psi_0) \tag{4b}$$

on the lower side of the wedge, $\omega' < 0$, and $\bar{n}' = -\bar{\imath}\sin\omega' + \bar{\jmath}\cos\omega'$. Then

$$q_n' = -U\sin\omega' \tag{5a}$$

$$B_n' = -B_0\sin(\omega' - \psi_0) \tag{5b}$$

Substituting Eqs. (4a–5b) into (2), one gets

$$\sin^4\omega - (M^{-2} + A^{-2})\sin^2\omega + M^{-2}A^{-2}\sin^2(\omega - \psi_0) = 0 \tag{6}$$

where $M^2 = U^2/a_0^2$ is the freestream Mach number squared. Equation (6) gives two values for ω and two values for ω'. They are all real for the super-fast hyperbolic flow considered in this paper, and they correspond to the angles made with the x axis by the four wave fronts, one fast and one slow on each side of the wedge (Fig. 4). Moreover, one has, always,

$$\omega_f > \omega_s > 0 \qquad \omega_f' < \omega_s' < 0$$

4. Solution of the Problem for Arbitrary ψ_0

Substituting Eqs. (4a–5b) into (3a) and (3b), one obtains, across each of the four wave fronts, the following:

$$\delta\bar{B}_f = K_fB_0\sin^2\omega_f\cos(\omega_f - \psi_0)(\bar{\imath}\cos\omega_f + \bar{\jmath}\sin\omega_f) \tag{7a}$$

$$\delta\bar{B}_s = -K_sB_0\sin^2\omega_s\cos(\omega_s - \psi_0)(\bar{\imath}\cos\omega_s + \bar{\jmath}\sin\omega_s) \tag{7b}$$

$$\delta\bar{B}_f' = K_f'B_0\sin^2\omega_f'\cos(\omega_f' - \psi_0)(\bar{\imath}\cos\omega_f' + \bar{\jmath}\sin\omega_f') \tag{7c}$$

$$\delta\bar{B}_s' = -K_s'B_0\sin^2\omega_s'\cos(\omega_s' - \psi_0)(\bar{\imath}\cos\omega_s' + \bar{\jmath}\sin\omega_s') \tag{7d}$$

and

$$\delta\bar{q}_f = K_fU\sin\omega_f\left\{\bar{\imath}\left[\frac{\cos\psi_0\sin(\omega_f - \psi_0)}{A^2} - \sin^3\omega_f\right] + \bar{\jmath}\left[\frac{\sin\psi_0\sin(\omega_f - \psi_0)}{A^2} + \sin^2\omega_f\cos\omega_f\right]\right\} \tag{8a}$$

$$\delta\bar{q}_s = -K_sU\sin\omega_s\left\{\bar{\imath}\left[\frac{\cos\psi_0\sin(\omega_s - \psi_0)}{A^2} - \sin^3\omega_s\right] + \bar{\jmath}\left[\frac{\sin\psi_0\sin(\omega_s - \psi_0)}{A^2} + \sin^2\omega_s\cos\omega_s\right]\right\} \tag{8b}$$

$$\delta\bar{q}_f' = K_f'U\sin\omega_f'\left\{\bar{\imath}\left[\frac{\cos\psi_0\sin(\omega_f' - \psi_0)}{A^2} - \sin^3\omega_f'\right] + \bar{\jmath}\left[\frac{\sin\psi_0\sin(\omega_f' - \psi_0)}{A^2} + \sin^2\omega_f'\cos\omega_f'\right]\right\} \tag{8c}$$

$$\delta\bar{q}_s' = -K_s'U\sin\omega_s'\left\{\bar{\imath}\left[\frac{\cos\psi_0\sin(\omega_s' - \psi_0)}{A^2} - \sin^3\omega_s'\right] + \bar{\jmath}\left[\frac{\sin\psi_0\sin(\omega_s' - \psi_0)}{A^2} + \sin^2\omega_s'\cos\omega_s'\right]\right\} \tag{8d}$$

The boundary conditions are the continuity of the magnetic field through the wedge:

$$\delta\bar{B}_f + \delta\bar{B}_s = \delta\bar{B}_f' + \delta\bar{B}_s' \tag{9a}$$

and the tangency of the flow along each of the wedge surfaces:

$$\bar{\jmath}\cdot\{\delta\bar{q}_f + \delta\bar{q}_s\} = U_0\theta \quad (9b)$$

$$\bar{\jmath}\cdot\{\delta\bar{q}_f' + \delta\bar{q}_s'\} = U_0\theta' \quad (9c)$$

Substitution of Eqs. (7a–7d and 8a–8d) into Eqs. (9a) and (6c) yields the following system of equations:

$$K_f \sin^2\omega_f \cos\omega_f \cos(\omega_f - \psi_0) - K_s \sin^2\omega_s \cos\omega_s \cos(\omega_s - \psi_0) - K_f' \sin^2\omega_f' \cos\omega_f' \cos(\omega_f' - \psi_0) + K_s' \sin^2\omega_s' \cos\omega_s' \cos(\omega_s' - \psi_0) = 0 \quad (10a)$$

$$K_f \sin^3\omega_f \cos(\omega_f - \psi_0) - K_s \sin^3\omega_s \cos(\omega_s - \psi_0) - K_f' \sin^3\omega_f' \cos(\omega_f' - \psi_0) + K_s' \sin^3\omega_s' \cos(\omega_s' - \psi_0) = 0 \quad (10b)$$

$$K_f\left[\frac{\sin\psi_0 \sin(\omega_f - \psi_0)}{A^2} + \sin^2\omega_f \cos\omega_f\right] - K_s\left[\frac{\sin\psi_0 \sin(\omega_s - \psi_0)}{A^2} + \sin^2\omega_s \cos\omega_s\right] = \theta \quad (10c)$$

$$K_f'\left[\frac{\sin\psi_0 \sin(\omega_f' - \psi_0)}{A^2} + \sin^2\omega_f' \cos\omega_f'\right] - K_s'\left[\frac{\sin\psi_0 \sin(\omega_s' - \psi_0)}{A^2} + \sin^2\omega_s' \cos\omega_s'\right] = \theta' \quad (10d)$$

from which the solution of the strengths K_f, K_s, K_f', and K_s' are obtained explicitly as

$$K_f = (E_f/\Delta)\{[D_f' \sin(\omega_s - \omega_s') - D_s' \sin(\omega_s - \omega_f')]\theta + D_s \sin(\omega_s' - \omega_f')\theta'\} \quad (11a)$$

$$K_s = (E_s/\Delta)\{[D_f' \sin(\omega_f - \omega_s') - D_s' \sin(\omega_f - \omega_f')]\theta + D_f \sin(\omega_s' - \omega_f')\theta'\} \quad (11b)$$

$$K_f' = (E_f'/\Delta)\{D_s' \sin(\omega_f - \omega_s)\theta + [D_f \sin(\omega_s - \omega_s') - D_s \sin(\omega_f - \omega_s')]\theta'\} \quad (11c)$$

$$K_s' = (E_s'/\Delta)\{D_f' \sin(\omega_f - \omega_s)\theta + [D_f \sin(\omega_s - \omega_f') - D_s \sin(\omega_f - \omega_f')]\theta'\} \quad (11d)$$

where

$$D_f = D_f(\omega_f) = \frac{\sin\psi_0 \tan(\omega_f - \psi_0)}{A^2 \sin\omega_f} + \frac{\sin\omega_f \cos\omega_f}{\cos(\omega_f - \psi_0)}$$

similarly for $D_s(\omega_s)$, $D_f'(\omega_f')$, and $D_s'(\omega_s')$,

$$E_f = E_f(\omega_f) = \frac{1}{\sin^2\omega_f \cos(\omega_f - \psi_0)}$$

similarly for $E_s(\omega_s)$, $D_f'(\omega_f')$, and $D_s'(\omega_s')$, and

$$\Delta = D_f D_f' \sin(\omega_s - \omega_s') - D_s D_f' \sin(\omega_f - \omega_s') - D_f D_s' \sin(\omega_s - \omega_f') + D_s D_s' \sin(\omega_f - \omega_f')$$

Once the four strength parameters K_f, K_s, K_f', K_s' are obtained, the change of each variable $\delta\rho$, $\delta\bar{B}$, etc., across each wave is given by (3a–3c), and the state of each uniform region is determined completely.

5. Crossed-Field Case: Physical Interpretation

The solution given in the previous paragraph is rather cumbersome and not readily amenable to interpretation. To obtain a clearer physical picture, the special case of a crossed-field flow, in which $\psi_0 = \pi/2$, will be examined in detail. The solution greatly simplifies, and some interesting features can be brought out explicitly. In this case, by symmetry,

$$\omega_f' = -\omega_f < 0 \qquad \omega_s' = -\omega_s < 0$$

Equation (6) becomes simply

$$\sin^4\omega - (M^{-2} + A^{-2} + M^{-2}A^{-2}) \sin^2\omega + M^{-2}A^{-2} = 0 \quad (12a)$$

hence

$$\sin\omega_f = -\sin\omega_f' = \tfrac{1}{2}\{M^{-2} + A^{-2} + M^{-2}A^{-2} + [(M^{-2} + A^{-2} + M^{-2}A^{-2})^2 - 4M^{-2}A^{-2}]^{1/2}\}^{1/2} \quad (12b)$$

and

$$\sin\omega_s = -\sin\omega_s' = \tfrac{1}{2}\{M^{-2} + A^{-2} + M^{-2}A^{-2} - [(M^{-2} + A^{-2} + M^{-2}A^{-2})^2 - 4M^{-2}A^{-2}]^{1/2}\}^{1/2} \quad (12c)$$

Without loss of generality, it will be assumed throughout that the upstream sound speed is greater than the upstream Alfven speed; hence $M^2 < A^2$. Also, the hyperbolicity in this case is given by $M^{-2} + A^{-2} < 1$, corresponding to $\sin^2\omega_f < 1$ (cf. also McCune and Resler[11]).

For brevity, the following notation is introduced:

$$\varphi_f = \sin^2\omega_f - A^{-2} \qquad \varphi_s = \sin^2\omega_s - A^{-2}$$

$$G = (\varphi_f \sin^2\omega_s - \varphi_s \sin^2\omega_f)(\varphi_f \sin^3\omega_s \cos\omega_f - \varphi_s \sin^3\omega_f \cos\omega_s)$$

The solution given by (11a–11d) then simplifies to

$$K_f = C_1\theta - C_2\theta' \quad (13a)$$

$$K_s = C_3\theta + C_4\theta' \quad (13b)$$

$$K_f' = C_2\theta - C_1\theta' \quad (13c)$$

$$K_s' = -(C_4\theta - C_3\theta') \quad (13d)$$

where

$$C_1 = \frac{\sin^2\omega_s}{G \sin 2\omega_f}[2\varphi_f \sin^3\omega_s \cos\omega_f - \varphi_s \sin^2\omega_f \sin(\omega_f + \omega_s)]$$

$$C_2 = \frac{-\varphi_s \sin^2\omega_f \sin^2\omega_s \sin(\omega_f - \omega_s)}{G \sin 2\omega_f}$$

$$C_3 = \frac{\sin^2\omega_f}{G \sin 2\omega_s}[-2\varphi_s \sin^3\omega_f \cos\omega_s + \varphi_f \sin^2\omega_s \sin(\omega_f + \omega_s)]$$

$$C_4 = \frac{\varphi_f \sin^2\omega_f \sin^2\omega_s}{G \sin 2\omega_s}\sin(\omega_f - \omega_s)$$

It is readily seen that $\varphi_f > 0$ and $\varphi_s < 0$:

$$\begin{aligned}\varphi_f = \sin^2\omega_f - A^{-2} &= \tfrac{1}{2}\{M^{-2} - A^{-2} + M^{-2}A^{-2} + [(M^{-2} + A^{-2} + M^{-2}A^{-2})^2 - 4M^{-2}A^{-2}]^{1/2}\}\\ &= \tfrac{1}{2}\{M^{-2} - A^{-2} + M^{-2}A^{-2} + [(M^{-2} - A^{-2})^2 + 2M^{-2}A^{-2} \times (M^{-2} + A^{-2})]^{1/2} + M^{-4}A^{-4}\}\\ &> \tfrac{1}{2}\{M^{-2} - A^{-2} + M^{-2}A^{-2} + [(M^{-2} - A^{-2})^2]^{1/2}\}\\ &= M^{-2}(1 - M^2/A^2) + \tfrac{1}{2}M^{-2}A^{-2}\\ &> 0\end{aligned}$$

since, by hypothesis, $M/A < 1$. Similarly,

$$\varphi_s = \sin^2\omega_s - A^{-2} = \tfrac{1}{2}\{M^{-2} - A^{-2} + M^{-2}A^{-2} - [(M^{-2} + A^{-2} + M^{-2}A^{-2})^2 - 4M^{-2}A^{-2}]^{1/2}\}$$
$$= \tfrac{1}{2}\{M^{-2} - A^{-2} + M^{-2}A^{-2} - [(M^{-2} - A^{-2} + M^{-2}A^{-2})^2 + 4M^{-2}A^{-4}]^{1/2}\}$$
$$< 0$$

Using this and the fact that $\omega_f > \omega_s > 0$, one immediately concludes that $G > 0$ and that all the constants C_1, C_2, C_3, and C_4 are > 0. Moreover, one has $C_1 > C_2$ and $C_3 > C_4$. These facts are of importance in interpreting the results, since the quantity in the square root obviously is greater than the quantity outside.

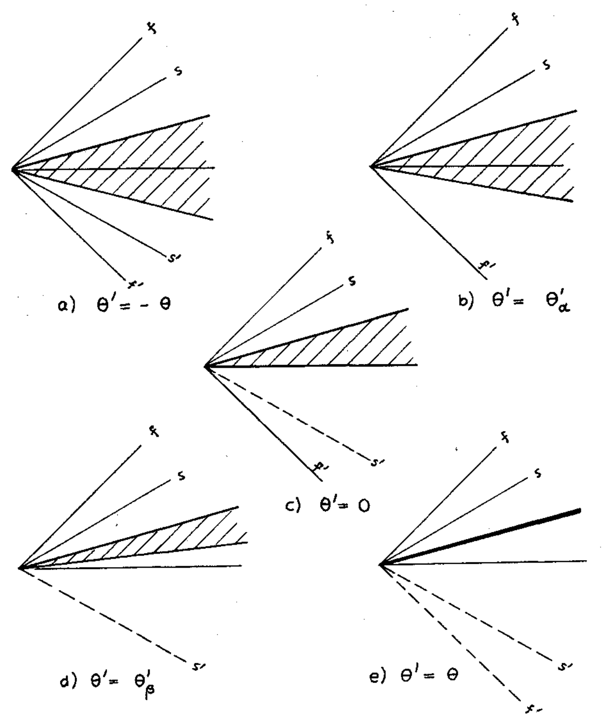

Fig. 5 Flow and wave patterns for different wedge angles θ' with θ fixed; upstream flow perpendicular to magnetic field, $\psi_0 = \pi/2$ (——— = compression, - - - - = expansion)

It will now be shown how the waves vary as the angles θ and θ' change. To this end, a symmetrical wedge with $\theta' = -\theta$ will first be considered, and then different wedges having the same fixed angle θ but different angles θ' $(-\theta \leq \theta' \leq \theta)$. The waves will be identified as shocks or expansions, and the way in which their strengths vary will be shown.

1) Symmetric wedge (Fig. 5a) $\theta' = -\theta$: From (13a–13d), one sees that $K_f = K_f' = (C_1 + C_2)\,\theta > 0$, and $K_s = K_s' = (C_3 - C_4)\theta > 0$, i.e., all the waves are shocks; the two fast shocks are equal in strength, and the two slow shocks are equal in strength.

2) Disappearance of the lower slow shock (Fig. 5b): As θ' is increased, a value $\theta' = \theta_\alpha'$ is reached at which $-\theta < \theta_\alpha' = -(C_4/C_3)\theta < 0$, and K_s' vanishes. This corresponds to a wedge configuration for which, at the flow variables given, there is only one wave underneath. This is a fast shock, since K_f' is still positive, and its strength is lower than that in case 1. A further increase in θ' results in a slow expansion wave below.

3) Flat lower surface (Fig. 5c) $\theta' = 0$: In this case, $K_f = C_1\theta$, $K_s = C_3\theta$, $K_f' = C_2\theta$, and $K_s' = -C_4\theta$. The slow wave below is an expansion wave, whereas all the other waves are shocks. This is in contradistinction to gasdynamics, in which the lower flow is undisturbed.

4) Disappearance of the lower shock (Fig. 5d): As θ' is increased to θ_β', $0 < \theta_\beta' = (C_2/C_1)\theta < \theta$, the lower fast shock disappears, leaving a single slow expansion wave below.

5) Flat plate (Fig. 5e) $\theta' = \theta$: In this case, $K_f = -K_f' = (C_1 - C_2)\theta > 0$, and $K_s = -K_s' = (C_3 + C_4)\theta$. There are two shocks on top and two expansion waves below. The fast shock and the fast expansion wave are equal in strength, and the slow shock and the slow expansion wave are equal in strength. This case coincides with that considered by McCune and Resler.[11]

It is also interesting to note that, as θ' is being increased from $-\theta$ to θ, the upper fast shock decreases in strength from $K_f = (C_1 + C_2)\theta$ to $K_f = (C_1 - C_2)\theta$, and the slow shock increases in strength from $K_s = (C_3 - C_4)\theta$ to $K_s = (C_3 + C_4)\theta$. Thus, all shocks weaken (and expansion waves strengthen) as θ' decreases, with the exception of the upper slow shock, which strengthens. A similar discussion will give the flow patterns around a given fixed-angled wedge at different angles of attack; however, this will now be done here.

The physical phenomena shown here can be described conveniently by a single parameter r defined by

$$r = (\theta + \theta')/(\theta - \theta')$$

It is equal to twice the ratio of the angle of attack of the wedge to the total wedge angle. Thus one obtains, for case 1 of symmetric wedge, $r_a = 0$. Similarly, for cases 2–5, one has

$$r_b = \frac{\varphi_f \sin^2\omega_s \sin2\omega_s \cos2\omega_f - \varphi_s \sin^3\omega_f \cos\omega_s}{\varphi_f \sin^2\omega_s \sin2\omega_f \cos2\omega_s - \varphi_s \sin^3\omega_f \cos\omega_s} < 1$$

$$r_c = 1$$

$$r_d = \frac{\varphi_f \sin^3\omega_s \cos\omega_f - \varphi_s \sin^2\omega_f \sin2\omega_s \cos2\omega_f}{\varphi_f \sin^3\omega_s \cos\omega_f - \varphi_s \sin^2\omega_f \sin2\omega_f \cos2\omega_s} > 1$$

$$r_e = \infty$$

r depends on the upstream field quantities only. One may conclude, on the lower side of the wedge, that 1) for $0 = r_a < r < r_b$, there are one fast and one slow shock; 2) for $r_b < r < r_d$, there are one fast shock and one slow expansion wave ($r_b < r = r_c < r_d$ corresponds to the case of a flat bottom); and 3) for $r_d < r < r_e < \infty$, there are one fast and one slow expansion wave.

This is valid for any wedge angle at any angle of attack, so long as they are both small and linearization is justified.

For large wedge angles, each small compression wave will be replaced by a finite shock, and each small expansion wave by an expansion fan. The flow patterns, however, will be qualitatively the same.

6. The Gasdynamic Limit

Now consider what happens to the flow in the crossed-field case as the magnetic field weakens to zero (i.e., $A \to \infty$), with U, a_0, θ, θ' held constant. Since linear theory holds for $\theta,\theta' \to 0$, i.e., for θ,θ' much smaller than all other occurring parameters, whereas it is now required that $1/A \to 0$, there is involved here an interchange of limiting processes (in general not permissible), and one should not expect meaningful results throughout.

It is readily seen by expansion that, as $1/A \to 0$,

$$\sin^2\omega_s = 0\left(\frac{1}{A^2}\right) \qquad \varphi_s = 0\left(\frac{1}{A^4}\right)$$

$$\sin^2\omega_f \to \frac{1}{M^2} \qquad \varphi_f \to \frac{1}{M^2}$$

$$G \to \frac{(M^2-1)^{1/2}}{M^5}\sin^5\omega_s = \frac{(M^2-1)^{1/2}}{M^5}\,0\left(\frac{1}{A^5}\right)$$

Thus it is seen from (18a) and (18c) that

$$K_f \to \frac{M^4}{(M^2 - 1)^{1/2}}\,\theta + 0\left(\frac{1}{A}\right)\theta'$$

$$K_f' \to -\frac{M^4}{(M^2 - 1)^{1/2}}\,\theta' + 0\left(\frac{1}{A}\right)\theta$$

The first terms are exactly the same as for weak shocks in ordinary gasdynamics. Hence, the fast shocks become ordinary gas shocks. The second terms indicate that the coupling between the upper and lower flows weakens toward zero as $A \to \infty$.

The slow waves also weaken (i.e., $\delta\rho_s/\rho \to 0$, etc.) as $A \to \infty$. However, as the angle ω_s tends to zero, θ and θ' must decrease correspondingly in order for linear theory to be applicable. If θ and θ' are held fixed, therefore, the result as $A \to \infty$ becomes meaningless. As already pointed out by Grad,[5] the slow wave collapses toward the body and becomes a thin region around it, in which the flow cannot be expected to tend to the gasdynamic limit. Such difficulties, of course, will not occur in the results of a nonlinear formulation.

Had $A < M$ been assumed instead of $A > M$, exactly the same behavior of the slow wave would occur as $M \to \infty$ and A remains fixed. Moreover, whether $M > A$ or $M < A$, analogous difficulties would occur with the fast wave as both M and A tend to infinity, but this is exactly similar to the hypersonic limit in ordinary gasdynamics, and one naturally should not expect the linear theory to hold.

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Analysis of the Fluid Mechanics of Secondary Injection for Thrust Vector Control

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An analysis is made of the interaction of an injected gas or liquid with a supersonic stream, and the force induced on an adjacent wall is predicted. The study deals only with the freestream-injectant interaction; the modifications to the flow introduced by the boundary layer are not considered. In the case of liquids, it is shown that the momentum deficit of the injectant relative to the freestream may play a larger part in producing the side force than the volume generation by vaporization and reaction. The analytical results are compared with those obtained from experiments in a wind tunnel and in nozzles.

Nomenclature

c = speed of sound
C_p = specific heat at constant pressure
E = energy per unit length
E_1 = energy per unit area
F_a = axial force or thrust, for exhaust to vacuum
F_i = interaction force
F_j = injection jet reaction force
F_m = maximum interaction force
F_w = wall force
F_r = axial force or thrust
F_s = side force
F_v = injection jet reaction force, for exhaust to vacuum
g = nondimensional pressure ratio
$\bar{g}$ = average nondimensional pressure ratio
$(I_{sp})_s$ = side specific impulse
J_0 = nondimensional constant
J_1 = nondimensional constant
K = amplification ratio, $(F_s/m_i)/(F_a/m_p)$
L = pressure field length
M = Mach number
m = mass flow
n = molecular weight
p = pressure
p_T = total pressure
r = radial distance
R = shock wave radius
$\bar{R}$ = gas constant
t = time
T = temperature

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